Definable Regularity for NIP Relations

Roland Walker

University of Illinois at Chicago

April 19, 2017

(Simple) Graphs

Let V be a finite set, and let $E \subseteq V \times V$.

Definition

If E is irreflexive and symmetric, we call $G = (V, E)$ a *(simple) graph* with

- *vertex set* $V(G) := V$
- *order* $v(G) := |V|$
- *edge set* $E(G) := \{\{a, b\} : (a, b) \in E\}$
- *size* $e(G) := |E(G)| = |E|/2$

Bipartite Graphs

Let V and W be finite sets, and let $E \subseteq V \times W$.

Definition

We say that $G = (V, W; E)$ is a *bipartite graph* with

- *vertex set* $V(G) := V + W$
- *order* $v(G) := |V| + |W|$
- *edge set* $E(G) := E$
- *size* $e(G) := |E|$

Note: Any graph $G = (V, E)$ induces a bipartite graph $G' = (V, V; E)$.

Edge Density for Induced Bipartite Graphs

Let $G = (V, E)$ be a finite graph.

Definition

Given $A, B \subseteq V$, the induced bipartite graph $(A, B; E)$ has

- *vertex set* $V(A, B) := A + B$
- *order* $v(A, B) := |A| + |B|$
- *edge set* $E(A, B) := E \cap (A \times B)$
- *size* $e(A, B) := |E(A, B)|$
- *density* $d(A, B) := \dfrac{e(A, B)}{|A||B|}$

Note: We do not require A and B to be disjoint.

Regularity and Defect

Let $G = (V, E)$ be a finite graph. Fix $\varepsilon, \delta \in [0, 1]$.

Definition

Given $A, B \subseteq V$, we say the pair (A, B) is (ε, δ)-*regular* iff: there exists $\alpha \in [0, 1]$ such that for all nonempty sets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \delta|A|$ and $|B'| \geq \delta|B|$, we have $|d(A', B') - \alpha| \leq \frac{\varepsilon}{2}$.

Let P be a finite partition of V. Fix $\eta \in [0, 1]$.

Definition

The *defect* of P is

$$\mathrm{def}_{\varepsilon,\delta}(P) := \{(A, B) \in P^2 : (A, B) \text{ not } (\varepsilon, \delta)\text{-regular}\},$$

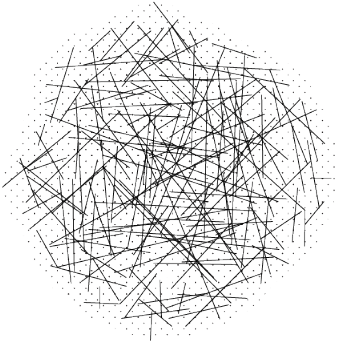
and we say that P is $(\varepsilon, \delta, \eta)$-regular iff:

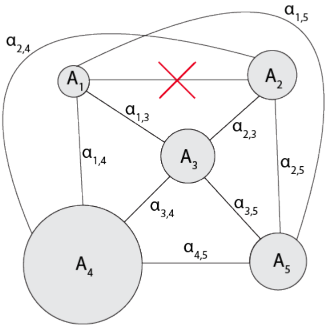
$$\sum_{(A,B) \,\in\, \mathrm{def}_{\varepsilon,\delta}(P)} |A||B| \leq \eta|V|^2.$$

Szemerédi Regularity Lemma (without Equipartition)

Lemma

For all $\varepsilon, \delta, \eta > 0$, there exists $M = M(\varepsilon, \delta, \eta)$ such that any finite graph has an $(\varepsilon, \delta, \eta)$-regular partition with at most M parts.





Szemerédi Regularity Lemma (without Equipartition)

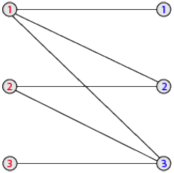
Lemma

For all $\varepsilon, \delta, \eta > 0$, there exists $M = M(\varepsilon, \delta, \eta)$ such that any finite graph has an $(\varepsilon, \delta, \eta)$-regular partition with at most M parts.

(Szemerédi 1976) $M(\varepsilon, \varepsilon, \varepsilon) \leq \mathrm{twr}_2(O(\varepsilon^{-5}))$

Can irregular pairs be completely eliminated?

No, if we admit arbitrarily large half-graphs, then irregular pairs are necessary.



Szemerédi Regularity Lemma (without Equipartition)

Lemma

For all $\varepsilon, \delta, \eta > 0$, there exists $M = M(\varepsilon, \delta, \eta)$ such that any finite graph has an $(\varepsilon, \delta, \eta)$-regular partition with at most M parts.

(Szemerédi 1976) $M(\varepsilon, \varepsilon, \varepsilon) \leq \mathrm{twr}_2(O(\varepsilon^{-5}))$

Can irregular pairs be completely eliminated?

No, if we admit arbitrarily large half-graphs, then irregular pairs are necessary.

<u>How fast does M grow as $\delta \to 0$?</u>

(Gowers 1997) $M(1 - \delta^{1/16}, \delta, 1 - 20\delta^{1/16}) \geq \mathrm{twr}_2(\Omega(\delta^{-1/16}))$

<u>How fast does M grow as $\eta \to 0$?</u>

(Conlon-Fox 2012) $\exists\ \varepsilon, \delta > 0$ such that $M(\varepsilon, \delta, \eta) \geq \mathrm{twr}_2(\Omega(\eta^{-1}))$

Bipartite Regularity and Defect

Let $G = (V_1, V_2; E)$ be a finite bipartite graph.

Let $\overline{P} = (P_1, P_2)$ where each P_i is a finite partition of V_i.

We will use the $\|\overline{P}\|$ to denote $\max(|P_1|, |P_2|)$.

Let $\varepsilon, \delta, \eta \in [0, 1]$.

Definition

The *defect* of $\overline{P}$ is

$$\mathrm{def}_{\varepsilon,\delta}(\overline{P}) := \{(A, B) \in P_1 \times P_2 : (A, B) \text{ not } (\varepsilon, \delta)\text{-regular}\},$$

and we call $\overline{P}$ $(\varepsilon, \delta, \eta)$-regular iff:

$$\sum_{(A,B) \,\in\, \mathrm{def}_{\varepsilon,\delta}(\overline{P})} |A||B| \leq \eta |V_1||V_2|.$$

Bipartite Szemerédi Regularity Lemma

Lemma

For all $\varepsilon, \delta, \eta > 0$, there exists $M = M(\varepsilon, \delta, \eta)$ such that any finite bipartite graph has an $(\varepsilon, \delta, \eta)$-regular partition P with $\|P\| \leq M$.

(Gowers 1997) $\Rightarrow M(1 - \delta^{1/16}, \delta, \frac{8}{9} - 40\delta^{1/16}) \geq \mathrm{twr}_2(\Omega(\delta^{-1/16}))$

k-Partite k-Uniform Hypergraphs

Let $k \geq 2$, $V_1, \ldots, V_k$ be finite sets, and $E \subseteq V_1 \times \cdots \times V_k$.

Definition

We say that $G = (V_1, \cdots, V_k; E)$ is a *k-partite (k-uniform) hypergraph* with

- *vertex set* $V(G) := V_1 + \cdots + V_k$
- *order* $v(G) := |V_1| + \cdots + |V_k|$
- *edge set* $E(G) := E$
- *size* $e(G) := |E|$

Note: When $k = 2$, $G = (V_1, V_2; E)$ is a bipartite graph.

Edge Density for k-Partite Hypergraphs

Let $G = (V_1, \ldots, V_k; E)$ be a finite k-partite hypergraph.

Definition

Given $A_i \subseteq V_i$, the subgraph $(A_1, \ldots, A_k; E)$ has

- *vertex set* $V(A_1, \ldots, A_k) := A_1 + \cdots + A_k$
- *order* $v(A_1, \ldots, A_k) := |A_1| + \cdots + |A_k|$
- *edge set* $E(A_1, \ldots, A_k) := E \cap (A_1 \times \cdots \times A_k)$
- *size* $e(A_1, \ldots, A_k) := |E(A_1, \ldots, A_k)|$
- *density* $d(A_1, \ldots, A_k) := \dfrac{e(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}$

Rectangular Sets

Definition

A tuple of sets $\overline{A} = (A_1, \ldots, A_k)$ names the *rectangular set* $A_1 \times \cdots \times A_k$.

- We write $B \subseteq \overline{A}$ iff: $B \subseteq A_1 \times \cdots \times A_k$.
- We write $\overline{B} \subseteq \overline{A}$ iff: each $B_i \subseteq A_i$.
- We use $|\overline{A}|$ to denote $|A_1| \cdots |A_k|$.
- We use $\|\overline{A}\|$ to denote $\max(|A_i| : 1 \leq i \leq k)$.

k-Partite Regularity and Defect

Let $G = (V_1, \ldots, V_k; E)$ be a finite k-partite hypergraph.

Fix $\varepsilon, \delta, \eta \in [0, 1]$.

Definition

Given $\overline{A} \subseteq \overline{V}$, we say $\overline{A}$ is (ε, δ)-*regular* iff: there exists $\alpha \in [0, 1]$ such that for all nonempty $\overline{B} \subseteq \overline{A}$ with $|B_i| \geq \delta |A_i|$, we have $|d(\overline{B}) - \alpha| \leq \frac{\varepsilon}{2}$.

Let $\overline{P} = (P_1, \ldots, P_k)$ where each P_i is a finite partition of V_i.

Definition

The *defect* of $\overline{P}$ is

$$\mathrm{def}_{\varepsilon,\delta}(\overline{P}) := \{\overline{A} \in \overline{P} : \overline{A} \text{ not } (\varepsilon, \delta)\text{-regular}\},$$

and we call $\overline{P}$ $(\varepsilon, \delta, \eta)$-regular iff:

$$\sum_{(A,B) \in \mathrm{def}_{\varepsilon,\delta}(P)} |\overline{A}| \leq \eta |\overline{V}|.$$

Fibers and VC Dimension

Let $E \subseteq V_1 \times \cdots \times V_k$.

For each $I \subseteq [k]$, let V_I denote $\prod_{i \in I} V_i$.

With $I \subseteq [k]$ specified, we can view E as a subset of $V_I \times V_{I^c}$ and for each $b \in V_{I^c}$, let E_b denote the fiber of b; i.e.,

$$E_b := \{a \in V_I : (a, b) \in E\}.$$

Definition

$\mathrm{VC}(E) = \max\{\mathrm{VC}(\mathcal{S}_I) : I \subseteq [k]\}$ where $\mathcal{S}_I = \{E_b : b \in V_{I^c}\}$.

Regularity Lemma for k-Partite Hypergraphs

Fix $k \geq 2$ and $d \in \mathbb{N}$.

Lemma

For any $\varepsilon, \delta, \eta > 0$, there is a constant $c = c(k, d)$ such that any finite k-partite hypergraph with VC *dimension at most d has an $(\varepsilon, \delta, \eta)$-regular partition $\overline{P}$ with $\|\overline{P}\| \leq O(\gamma^{-c})$ where $\gamma = \min\{\varepsilon, \delta, \eta\}$.*

Note: When $k = 2$, this is the Bipartite Szemerédi Regularity Lemma restricted to graphs with $\mathrm{VC}(E) \leq d$.

Finitely Additive Probability Measures

Let X be a set, $\mathcal{B} \subseteq \mathcal{P}(X)$ a boolean algebra, and $\mu : \mathcal{B} \to [0, 1]$.

Definition

We call μ a *finitely additive probability measure* iff:

- $\mu(\varnothing) = 0$
- $\mu(X) = 1$
- For all disjoint $A, B \in \mathcal{B}$, $\mu(A \cup B) = \mu(A) + \mu(B)$

For this talk, assume all measures are finitely additive probability measures.

Definition

If $\mathcal{M}$ is a model, we call a finitely additive probability measure on the boolean algebra of all definable subsets of M^n a *Keisler measure*.

Finitely Approximated Measures

Let X be a set, $\mathcal{B} \subseteq \mathcal{P}(X)$ be a boolean algebra.

Definition

For any finite sequence $\overline{p}$ in X of length $n \geq 1$, let $\mathrm{Fr}_{\overline{p}}$ denote the *frequency measure* determined by $\overline{p}$; i.e., for $B \in \mathcal{B}$, we have

$$\mathrm{Fr}_{\overline{p}}(B) = \frac{1}{n} \sum_{i=1}^{n} 1_B(p_i).$$

Let μ be a measure on $\mathcal{B}$.

Definition

For $\mathcal{F} \subseteq \mathcal{B}$, we say μ is *finitely approximated (fap)* on $\mathcal{F}$ iff: for all $\varepsilon > 0$, there is an ε-approximation $\overline{p} \in X$ such that for all $A \in \mathcal{F}$

$$|\mu(A) - \mathrm{Fr}_{\overline{p}}(A)| < \varepsilon.$$

E-Definable Sets

Let $E \subseteq V_1 \times \cdots \times V_k$ and $I \subseteq [k]$.

Definition

A subset of V_I is *E-definable* over $D \subseteq V_{I^c}$ iff: it is a boolean combination of sets of the form E_b for $b \in D$.

- We use $\mathcal{B}_E(D)$ to denote the boolean algebra of all such sets.
- If D is finite, we use $\mathcal{A}_E(D)$ to denote the atoms in $\mathcal{B}_E(D)$.

Density for Definable Rectangular Sets

Let $\mathcal{M}$ be a structure and $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$.

Let each $V_i = M^{|v_i|}$, $E = \phi(\overline{V})$, and $G = (\overline{V}; E)$.

Let each μ_i be a Keisler measure on V_i.

Definition

We say μ_i is *fap on E* iff: for all $n \in \mathbb{N}$, μ_i is fap on

$$\bigcup \left\{ \mathcal{B}_E(D) : D \subseteq V_{\{i\}^c} \text{ and } |D| \leq n \right\}.$$

Suppose each μ_i is fap on E, and let $\mu = \mu_1 \ltimes \cdots \ltimes \mu_k$.

It follows that μ is fap on E and satisfies a weak Fubini property.

Definition

The *density* of a definable $\overline{A} \subseteq \overline{V}$ is

$$d(\overline{A}) := \frac{\mu(E \cap \overline{A})}{\mu(\overline{A})} = \frac{\mu(\phi(\overline{A}))}{\mu_1(A_1) \cdots \mu_k(A_k)}.$$

Definable Regularity and Defect with 0-1 Densities

Fix $\varepsilon, \delta, \eta \in [0, 1]$.

Definition

Given definable $\overline{A} \subseteq \overline{V}$, we say $\overline{A}$ is (ε, δ)-*regular with 0-1 densities* iff: there exists $\alpha \in \{0, 1\}$ such that for all nonempty definable $\overline{B} \subseteq \overline{A}$ with $\mu(\overline{B}) \geq \delta\mu(\overline{A})$, we have $|d(\overline{B}) - \alpha| \leq \varepsilon$.

Let $\overline{P}$ be a partition of $\overline{V}$.

Definition

The *0-1 defect* of $\overline{P}$ is

$$\operatorname{def}^{\text{0-1}}_{\varepsilon,\delta}(\overline{P}) := \{\overline{A} \in \overline{P} : \overline{A} \text{ not } (\varepsilon, \delta)\text{-regular with 0-1 densities}\},$$

and we say $\overline{P}$ is $(\varepsilon, \delta, \eta)$-regular with 0-1 densities iff:

$$\sum_{\overline{A} \in \operatorname{def}^{\text{0-1}}_{\varepsilon,\delta}(\overline{P})} \mu(\overline{A}) \leq \eta.$$

Definable Regularity Lemma for NIP Relations

Let $k \geq 2$ and $d \in \mathbb{N}$.

Theorem

There is a constant $c = c(k, d)$ such that IF

- $\varepsilon, \delta, \eta > 0$
- $E = \phi(\overline{V})$ *for some* $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$ *and structure* $\mathcal{M}$
- $\mathrm{VC}(E) \leq d$
- *each* μ_i *is a Keisler measure on* V_i *which is fap on* E

THEN there is an $(\varepsilon, \delta, \eta)$*-regular partition* $\overline{P}$ *of* $\overline{V}$ *with* 0-1 *densities such that*

- $\|\overline{P}\| \leq O(\gamma^{-c})$ *where* $\gamma = \min\{\varepsilon, \delta, \eta\}$
- *for each* P_i*, the parts of* P_i *are definable using a single formula* ψ_i *which is a boolean combination of* ϕ *depending only on* γ *and* ϕ*.*

E-Definable Sets

Let $k \geq 2$.

Let $\mathcal{M}$ be a structure and $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$.

Let each $V_i = M^{|v_i|}$, $E = \phi(\overline{V})$, and $G = (\overline{V}; E)$.

Definition

A subset of V_I is *E-definable* over $D \subseteq V_{I^c}$ iff: it is a boolean combination of sets of the form E_b for $b \in D$.

- We use $\mathcal{B}_E(D)$ to denote the boolean algebra of all such sets.
- If D is finite, we use $\mathcal{A}_E(D)$ to denote the atoms in $\mathcal{B}_E(D)$.

Definition

A subset of $\overline{V}$ is *$E_\otimes$-definable* iff: it is a finite union of rectangular sets of the form $\overline{A} \subseteq \overline{V}$ where each A_i is E-definable.

Counting Atoms

Lemma

If $\mathrm{VC}(E) \leq d$ *and* $|D| = n$*, both finite, then*

$$\mathcal{A}_E(D) \leq \binom{n}{d} + \cdots + \binom{n}{0} \leq (d+1)n^d.$$

Proof: Sauer-Shelah. □

ε-Nets

Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a boolean algebra and μ a measure on $\mathcal{B}$.

Let $\varepsilon > 0$ and $\mathcal{F} \subseteq \mathcal{B}$.

Definition

We say $T \subseteq X$ is an *ε-net for $\mathcal{F}$* iff:
all sets $A \in \mathcal{F}$ with $\mu(A) \geq \varepsilon$ intersect T.

Lemma

If μ has finite support and $\mathrm{VC}(\mathcal{F}) \leq d$*, then for any $\varepsilon > 0$, there is an ε-net T for $\mathcal{F}$ such that*

$$|T| \leq \frac{8d}{\varepsilon} \log \frac{1}{\varepsilon}.$$

Using Counting Techniques

Let $k \geq 2$.

Let $\mathcal{M}$ be a structure, and let $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$ be NIP.

Let each $V_i = M^{|v_i|}$, $E = \phi(\overline{V})$, $d = \mathrm{VC}(E)$, and $G = (\overline{V}; E)$.

Let each μ_i be a Keisler measure on V_i which is fap on E.

Lemma (2.17)

If $\varepsilon > 0$, there exists $D_1 \subseteq V_{\{1\}^c}$ of size at most $320 d\varepsilon^{-2}$ such that for all $X \in \mathcal{A}_E(D_1)$ and all $a, a' \in X$, we have

$$\mu_{\{1\}^c}(E_a \triangle E_{a'}) < \varepsilon.$$

Proof of Lemma 2.17

Let $\mathcal{F} = \{E_a \triangle E_{a'} : a, a' \in V_1\} \subseteq \mathcal{P}(V_{\{1\}^c})$.

Since $\mu_{\{1\}^c}$ is fap on E, it has an $\frac{\varepsilon}{2}$-approximation $\overline{p}$ for $\mathcal{F}$.

Now $\mathrm{VC}(\mathcal{F}) \leq 10d$ and $\mathrm{Fr}_{\overline{p}}$ has finite support, so there is an $\frac{\varepsilon}{2}$-net $D_1 \subseteq V_{\{1\}^c}$ for $\mathcal{F}$ with

$$|D_1| \leq \frac{160d}{\varepsilon} \log \frac{2}{\varepsilon} \leq \frac{320d}{\varepsilon^2}.$$

Let $X \in \mathcal{A}_E(D_1)$ and $a, a' \in X$.

It follows that $E_a \cap D_1 = E_{a'} \cap D_1$, so $(E_a \triangle E_{a'}) \cap D_1 = \varnothing$.

Thus, $\mathrm{Fr}_{\overline{p}}(E_a \triangle E_{a'}) < \frac{\varepsilon}{2}$ and $\mu_{\{1\}^c}(E_a \triangle E_{a'}) < \varepsilon$. $\square$

Applying Fubini

Proposition (2.18)

If $\varepsilon > 0$*, there exists* $\overline{D} \subseteq (V_{\{1\}^c}, \ldots, V_{\{k\}^c})$ *and* $F \subseteq \overline{V}$ *which is* $E_\otimes$*-definable over* $\overline{D}$ *such that*

$$\mu(E \triangle F) \leq \varepsilon$$

and $\|\overline{D}\| \leq C\varepsilon^{-2d(k-1)}$ *where* $C = C(k, d)$*.*

Proof of Proposition 2.18

Let $D_1 \subseteq V_{\{1\}^c}$ be given by Lemma 2.17 for $\varepsilon/2$, so $|D_1| \leq 1280d\varepsilon^{-2}$.

Let $\{X_1, \ldots, X_m\}$ enumerate $\mathcal{A}_E(D_1)$.

Notice $V_1 = X_1 + \cdots + X_m$.

For each X_i, choose $a_i \in X_i$

Let $H = \bigsqcup_{i=1}^{m}(X_i \times E_{a_i})$.

Given $a \in V_1$, there is a unique atom X_i such that $a \in X_i$.

It follows that $H_a = E_{a_i}$, so $\mu_{\{1\}^c}(E_a \triangle H_a) < \varepsilon/2$ by Lemma 2.17.

Further, $(E \triangle H)_a = E_a \triangle H_a$, so $\mu(E \triangle H) \leq \varepsilon/2$ by Fubini.

<u>If $k = 2$:</u>

Let $F = H$ and $D_2 = \{a_i : i \in [m]\}$, so F is $E_{\otimes}$-definable over $\overline{D}$ and

$$m \leq (d+1)|D_1|^d \leq C(2,d)\varepsilon^{-2d}$$

where $C(2,d) = (d+1)(1280d)^d$.

Proof of Proposition 2.18 (cont'd)

If $k > 2$:

By induction, for each $i \in [m]$, we have a $Y_i \subseteq V_{\{1\}^c}$ which is $(E_{a_i})_\otimes$-definable over $(B_{i,2}, \ldots, B_{i,k})$ where

$$\|\overline{B_i}\| \le C(k-1,d)(\varepsilon/2)^{-2d(k-2)}$$

such that $\mu_{\{1\}^c}(E_{a_i} \triangle Y_i) \le \varepsilon/2$.

Let $F = \bigsqcup_{i=1}^m (X_i \times Y_i)$. Recall $H = \bigsqcup_{i=1}^m (X_i \times E_{a_i})$.
It follows that $F \triangle H = \bigsqcup_{i=1}^m (X_i \times (E_{a_i} \triangle Y_i))$, so $\mu(F \triangle H) \le \varepsilon/2$.

Further, $E \triangle F \subseteq (E \triangle H) \cup (H \triangle F)$, so $\mu(E \triangle F) \le \varepsilon$.

For $j \ge 2$, let $D_j = \bigcup_{i=1}^m B_{i,j}$.
Now F is $E_\otimes$-definable over $\overline{D}$ and

$$\|\overline{D}\| \le mC(k-1,d)(\varepsilon/2)^{-2d(k-2)} \le C(k,d)\varepsilon^{-2d(k-1)}$$

where $C(k,d) = 2^{2d(k-2)}C(2,d)C(k-1,d)$. $\square$

Let $\overline{P}$ be a rectangular partition of $\overline{V}$.

Definition

We say $F \subseteq \overline{V}$ is *compatible* with $\overline{P}$ iff: for all $\overline{A} \in \overline{P}$ either $\overline{A} \subseteq F$ or $\overline{A} \cap F = \varnothing$.

Definition

We call $\overline{A} \subseteq \overline{P}$ *ε-regular** iff: there exists $\alpha \in \{0, 1\}$ such that for all definable $\overline{B} \subseteq \overline{A}$, we have

$$|\mu(E(\overline{B})) - \alpha\mu(\overline{B})| \leq \varepsilon\mu(\overline{A}).$$

The defect* of $\overline{P}$ is

$$\text{def}^*_\varepsilon(\overline{P}) := \{\overline{A} \in \overline{P} : \overline{A} \text{ not } \varepsilon\text{-regular}^*\},$$

and we call $\overline{P}$ ε-regular* iff:

$$\sum_{\overline{A} \,\in\, \text{def}^*_\varepsilon(\overline{P})} \mu(\overline{A}) \leq \varepsilon.$$

Let $\overline{P}$ be a rectangular partition of $\overline{V}$.

Lemma (3.2)

If there exists an $E_{\otimes}$-definable $F \subseteq \overline{V}$ compatible with $\overline{P}$ such that $\mu(E \triangle F) \leq \varepsilon^2$, then $\overline{P}$ ε-regular.*

Proof: Let $\mathcal{D} = \left\{\overline{A} \in \overline{P} : \mu(\overline{A} \cap (E \triangle F)) > \varepsilon\mu(\overline{A})\right\}$, so

$$\sum_{\overline{A} \in \mathcal{D}} \mu(\overline{A}) \leq \varepsilon.$$

Let $\overline{A} \in \overline{P} \setminus \mathcal{D}$, and let $\overline{B}$ be a definable rectangular subset of $\overline{A}$.

It follows that $\mu(\overline{B} \cap (E \triangle F)) \leq \varepsilon\mu(\overline{A})$.

<u>Case 1: $\overline{A} \subseteq F$</u>

Since $\overline{B} \cap (E \triangle F) = \overline{B} \setminus E$, we have $\mu(\overline{B}) - \mu(E(\overline{B})) = \mu(\overline{B} \setminus E) \leq \varepsilon\mu(\overline{A})$.

<u>Case 2: $\overline{A} \cap F = \varnothing$</u>

Since $\overline{B} \cap (E \triangle F) = E(\overline{B})$, we have $\mu(E(\overline{B})) \leq \varepsilon\mu(\overline{A})$. $\square$

Proposition (3.3)

For any $\varepsilon > 0$, there is an E-definable ε-regular partition $\overline{P}$ with*

$$\|\overline{P}\| \leq (d+1)C(k,d)^d \varepsilon^{-4d^2(k-1)}.$$

Proof: By Proposition 2.18, there exists $\overline{D} \subseteq (V_{\{1\}^c}, \ldots, V_{\{k\}^c})$ with

$$\|\overline{D}\| \leq C(k,d)\varepsilon^{-4d(k-1)}$$

and an $F \subseteq \overline{V}$ which is $E_{\otimes}$-definable over $\overline{D}$ such that $\mu(E \triangle F) \leq \varepsilon^2$.

For each $i \in [k]$, let $P_i = \mathcal{A}_E(D_i)$, so $\|\overline{P}\| \leq (d+1)C(k,d)^d \varepsilon^{-4d^2(k-1)}$.

Let $\overline{A} \in \overline{P}$. Suppose $\overline{A} \cap F \neq \varnothing$, and let $\overline{a} \in \overline{A} \cap F$.

Since F is $E_{\otimes}$-definable, there exists $\overline{B} \subseteq F$ where each $B_i \in \mathcal{B}_E(D_i)$ such that $\overline{a} \in \overline{B}$.

It follows that $\overline{A} \subseteq \overline{B} \subseteq F$, so we can apply Lemma 3.2. □

Getting back to $(\varepsilon, \delta, \eta)$-Regularity

Lemma

If $\varepsilon, \delta, \eta > 0$, $\gamma = \min\{\varepsilon, \delta, \eta\}$ and $\overline{P}$ is γ^2-regular**, then $\overline{P}$ is $(\varepsilon, \delta, \eta)$-regular with 0-1 densities.*

Proof: Suppose $\overline{P}$ is γ^2-regular*, and let $\overline{A} \in \overline{P} \setminus \mathrm{def}^*_{\gamma^2}(\overline{P})$.

There is an $\alpha \in \{0, 1\}$ such that for all definable $\overline{B} \subseteq \overline{A}$, we have

$$|\mu(E \cap \overline{B}) - \alpha\mu(\overline{B})| \leq \gamma^2 \mu(\overline{A}).$$

It follows that

$$|d(\overline{B}) - \alpha| \leq \gamma^2 \frac{\mu(\overline{A})}{\mu(\overline{B})}$$

yielding

$$\mu(\overline{B}) < \gamma\mu(\overline{A}) \quad \text{or} \quad |d(\overline{B}) - \alpha| \leq \gamma.$$

Definable Regularity Lemma for NIP Relations

Let $k \geq 2$ and $d \in \mathbb{N}$.

Theorem

There is a constant $c = c(k, d)$ such that IF

- $\varepsilon, \delta, \eta > 0$
- $E = \phi(\overline{V})$ *for some* $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$ *and structure* $\mathcal{M}$
- $\mathrm{VC}(E) \leq d$
- *each* μ_i *is a Keisler measure on* V_i *which is fap on* E

THEN there is an $(\varepsilon, \delta, \eta)$*-regular partition* $\overline{P}$ *of* $\overline{V}$ *with* 0-1 *densities such that*

- $\|\overline{P}\| \leq O(\gamma^{-c})$ *where* $\gamma = \min\{\varepsilon, \delta, \eta\}$
- *for each* P_i*, the parts of* P_i *are definable using a single formula* ψ_i *which is a boolean combination of* ϕ *depending only on* γ *and* ϕ*.*

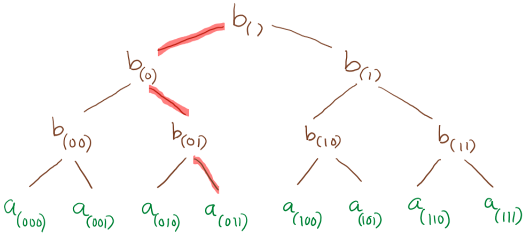
In particular, we showed $\|\overline{P}\| \leq (d+1)C(k, d)^d (1/\gamma)^{-8d^2(k-1)}$.

Stability

Let $d \in \mathbb{N}$ and $R \subseteq V \times W$.

Definition

We say R is d-stable iff: there is <u>not</u> a tree of parameters $\{b_\tau : \tau \in {}^{<d}2\} \subseteq W$ along with a set of leaves $\{a_\sigma : \sigma \in {}^{d}2\} \subseteq V$ such that for any $\sigma \in {}^{d}2$ and $n < d$, we have $(a_\sigma, b_{\sigma\restriction_n}) \in R \iff \sigma(n) = 1$.



$$a_{(011)} \models \neg R(x, b_{()}) \wedge R(x, b_{(0)}) \wedge R(x, b_{(01)})$$

Stability

Let $d \in \mathbb{N}$ and $R \subseteq V \times W$.

Definition

We say R is d-stable iff: there is not a tree of parameters $\{b_\tau : \tau \in {}^{<d}2\} \subseteq W$ along with a set of leaves $\{a_\sigma : \sigma \in {}^{d}2\} \subseteq V$ such that for any $\sigma \in {}^{d}2$ and $n < d$, we have $(a_\sigma, b_{\sigma\restriction_n}) \in R \iff \sigma(n) = 1$.

Let $k \geq 2$ and $E \subseteq V_1 \times \cdots \times V_k$.

Definition

We say E is d-stable iff: for all $I \subseteq [k]$, $E_I \times E_{I^c}$ is d-stable.

Definable Regularity Lemma for Stable Relations

Let $k \geq 2$ and $d \in \mathbb{N}$.

Theorem

There is a constant $c = c(k, d)$ such that IF

- $\varepsilon, \delta > 0$ *and* $\eta = 0$
- $E = \phi(\overline{V})$ *for some* $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$ *and structure* $\mathcal{M}$
- E *is* d*-stable*
- *each* μ_i *is a Keisler measure on* V_i *which is fap on* E

THEN there is an $(\varepsilon, \delta, \eta)$*-regular partition* $\overline{P}$ *of* $\overline{V}$ *with* 0-1 *densities such that*

- $\|\overline{P}\| \leq O(\gamma^{-c})$ *where* $\gamma = \min\{\varepsilon, \delta\}$
- *for each* P_i*, the parts of* P_i *are definable using a single formula* ψ_i *which is a boolean combination of* ϕ *depending only on* γ *and* ϕ.

Distality

Let T be a complete NIP theory and $\mathcal{U}$ a monster model for T.

Definition

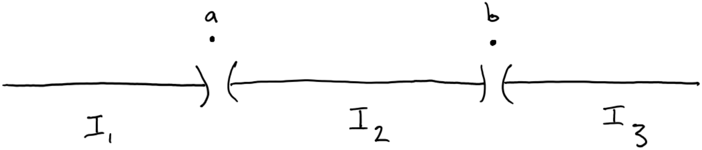
We say T is distal iff: for all $n \geq 1$, all indiscernible sequences $I \subseteq U^n$, and all Dedekind cuts $I = I_1 + I_2 + I_3$, if

$$I_1 + a + I_2 + I_3 \quad \text{and} \quad I_1 + I_2 + b + I_3$$

are both indiscernible, then

$$I_1 + a + I_2 + b + I_3$$

is also indiscernible.



Definable Regularity Lemma for Distal NIP Structures

Let T be a complete distal NIP theory and $\mathcal{M} \models T$.

Let $k \geq 2$ and $\phi(v_1, \ldots, v_k) \in \mathcal{L}_M$.

Theorem

There is a constant $c = c(\mathcal{M}, \phi)$ such that IF

- $\varepsilon = \delta = 0$ *and* $\eta > 0$
- $E = \phi(\overline{V})$
- *each μ_i is a Keisler measure on V_i which is fap on E*

THEN there is an $(\varepsilon, \delta, \eta)$-regular partition $\overline{P}$ of $\overline{V}$ with 0-1 densities such that

- $\|\overline{P}\| \leq O(\eta^{-c})$
- *for each P_i, the parts of P_i are definable using a single formula ψ_i which is a boolean combination of ϕ depending only on ϕ.*

References

[CF12] D. Conlon and J. Fox. Bounds for graph regularity and removal lemmas. *Geometric and Functional Analysis*, 22 (2012), 1191-1256.

[CS16] A. Chernikov and S. Starchenko. Definable regularity lemmas for NIP hypergraphs. arXiv:1607.07701 [math.LO].

[Gow97] W.T. Gowers. Lower bounds of tower type for Szemerédi's uniformity lemma. *Geometric and Functional Analysis*, 7 (1997), 322-337.

[KS96] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. In: *Combinatorics, Paul Erdõs is Eighty, Vol. 2* (Keszthely, 1993). *Bolyai Society Mathematical Studies, Vol. 2. János Bolyai Mathematical Society, Budapest* (1996), pp. 295-352.

[MS13] M. Malliaris and S. Shelah. Regularity lemmas for stable graphs. *Transactions of the American Mathematical Society*, Vol. 336, Number 3 (2013), 1551-1585.

[Sze76] E. Szemerédi. Regular partitions of graphs. In: Colloques Internationaux CNRS 260-*Problèmes Combinatoires et Théorie des Graphes*, Orsay (1976), pp. 399-401.